HEAT CONDUCTION WITH A TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

II. VALIDITY OF VARIATIONAL FORMALISM

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The nonlinear equation of heat conduction is solved by the method of finite differences. The results are compared with the solution to the same problem obtained by the variational method.

Introduction

Nonlinear differential equations can be solved by the variational method. This method has been used in several studies concerning heat conduction problems. Finlayson and Scriven in a very interesting and critical paper show that the variational principle proposed by Rosen [2], Chambers [3], and Biot for solving heat transfer problems represent special cases of the weighted-differences method [4]. In a more recent study, Finlayson and Scriven [5] analyze other attempts to formulate the variational principle and point out the error in these attempts stemming from the fact that the variation integral is nonstationary. Variational formalism is also discussed by Glansdorff, Prigogine, and Hays [6, 7]. Hays [8] and Hays and Curd [9] have formulated the variational principle by the "local potential" method. The latter method is based on the work by Prigogine in 1945, where reference is made, for instance, to the possibility of using the increase in entropy as a measure of "the approach of a physical system to equilibrium." For this rereason, the procedure in [5] is equivalent to the Galerkin method, which in turn represents one of five possible variants of the weighted-differences method. The paper by O'Toole [10] also deserves mention. Although the method outlined there is applicable only to special cases, it may serve as a basis for comparing the basis functions. In the same paper are given a few examples of problems solved by the variational method, from which it becomes evident that, even for multiparameter basis functions, the results obtained by various methods are as accurate as those obtained by less sophisticated schemes.

In this Part 2 of the article we will compare the variational method (specially formulated) with the finite-differences method; their merits and drawbacks will be analyzed on the example of the thin plate which had been considered in Part 1.

Main Contents. We use the same basis function as in Part 1:

$$\theta = \sum_n \left(e^{-\beta\lambda_n^2\tau} C_n + D_n\right) \cos \lambda_n \xi. \tag{1}$$

The temperature distribution can be expressed as a function of the dimensionless coordinate ξ at various times τ and for some values of σ, also for every q^* (for instance, we have determined distribution (1) with $q^* = 20$, 1, and 2.4674).

We will now analyze the same problem by the finite-differences method. The parabolic equation

$$[1 + \sigma\theta(\xi, \tau)] \frac{\partial^2\theta(\xi, \tau)}{\partial\xi^2} + \sigma\left[\frac{\partial\theta(\xi, \tau)}{\partial\xi}\right]^2 - \frac{\partial\theta(\xi, \tau)}{\partial\tau} + q^* = 0 \tag{2}$$

is solved by a well-known method. We will, therefore, outline the solution procedure schematically. The interval (0, 1) ξ-axis is subdivided into N equal steps $\Delta\xi = 1/N$. Let $\Delta\tau$ be called the integration step along

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TABLE 1. Steady Temperature Distributions Obtained by Integrating Eq. (2) according to the Runge–Kutta Method

q^*	σ	θ_{max}	$\theta(0, \infty)$	$\frac{\Delta\xi^2}{2+2\sigma\theta_{max}}$	$\Delta\tau$
	0,02	1,1	0,498	0,00122	0,001
1	0,1	1,1	0,489	0,00113	0,001
	1	1,1	0,414	0,00060	0,0005
	0,02	1,3	1,22	0,00122	0,001
2,4674	0,1	1,3	1,17	0,000605	0,0005
	1	1,3	0,862	0,000543	0,0005
	0,02	9,16	9,16	0,00106	0,001
20	0,1	7,32	7,32	0,00116	0,001
	1	3,58	3,58	0,00027	0,00025

the time axis. Let $\theta_{i,j}$ be the temperature at the instant of time $\tau = j\Delta\tau$ ($j = 0, 1, \ldots, \infty$) at point $\xi = i\Delta\xi$ ($i = -1, 0, 1, \ldots, N$). We approximate the derivatives in Eq. (2) as follows:

$$\left(\frac{\partial\theta}{\partial\tau}\right)_{i,j} = \frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta\tau} + 0(\Delta\tau), \tag{3}$$

$$\left(\frac{\partial^2\theta}{\partial\xi^2}\right)_{i,j} = \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta\xi)^2} + 0(\Delta\xi^2), \tag{4}$$

$$\left(\frac{\partial\theta}{\partial\xi}\right)_{i,j} = \frac{\theta_{i+1,j} - \theta_{i-1,j}}{2\Delta\xi} + 0(\Delta\xi^2). \tag{5}$$

In order to make the approximation error in (3) the same as in the other expressions, we let [11]

$$r = \frac{\Delta\tau}{(\Delta\xi)^2}. \tag{6}$$

Then Eq. (2) becomes

$$4r(1 + \sigma\theta_{i,j})(\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}) + \sigma r(\theta_{i+1,j} - \theta_{i-1,j})^2 - 4(\theta_{i,j+1} - \theta_{i,j}) + 4r(\Delta\xi)^2 q^* = 0, \tag{7}$$

without regard to approximation errors.

Equation (2) was solved for several values of σ and q^* (specifically, for $q^* = 20$, 1, and 2.4674). The stability condition

$$(1 + \sigma\theta)\, r < \frac{1}{2} \tag{8}$$

was found to be a very stringent one and it restricted considerably the choice of $\Delta\tau$ and $\Delta\xi$ steps. Indeed, from Eq. (8) we can find the maximum value of $r = \Delta\tau/(\Delta\xi)^2$. Evidently,

$$\left[\frac{\Delta\tau}{(\Delta\xi)^2}\right]_{max} = \frac{1}{2(1 + \sigma\theta_{max})}. \tag{9}$$

It is, therefore, necessary to determine θ_{max} at least approximately, which can be done owing to the physical nature of the problem.

The steady-state temperature distributions shown in Table 1 have been determined as a result of integrating Eq. (2) by the Runge–Kutta method. The table indicates that, in order to set the most appropriate step $\Delta\tau$, θ_{max} has been assumed 1.1 and 1.3 respectively for $q^* = 1$ and 2.4674 (these values have been selected on the basis of physical considerations). Integration of Eq. (7) by the method of finite differences has shown that these values are acceptable. Although computations were performed with various $\Delta\xi$ and $\Delta\tau$ steps satisfying (9), and by various finite-difference formulas of the (3)-(5) kind, expressions (3)-(5) and the choice of unit step $\Delta\xi = 0.05$ may be considered adequate for all computations in terms of the desired three-digit precision and in terms of machine time. The $\Delta\tau$ steps in the last column of the table correspond to the choice of $\Delta\xi$. If $\Delta\tau$ exceeds the respective corresponding value $(\Delta\xi)^2/(2 + 2\sigma\theta_{max})$ by even a few thousandths, instability results.

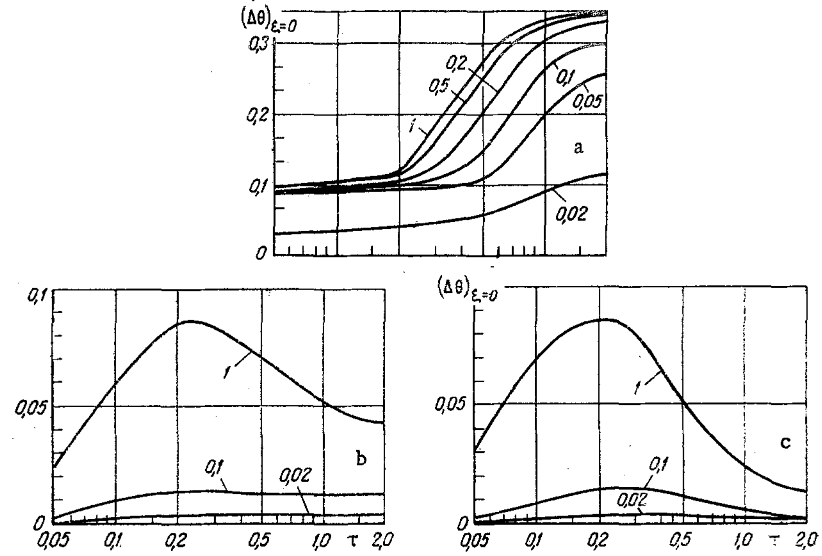

Fig. 1. Difference between calculated temperature distributions, as a function of time: a) $q^* = 20$; b) $q^* = 1$; c) $q^* = 2.4674$. Numbers at the curves indicate the values of σ.

DISCUSSION OF RESULTS

The difference between temperature distributions obtained by the variational method and by the finite-differences method $\Delta\theta$ is shown in Fig. 1a as a function of τ for several values of σ with $q^* = 20$. This difference was calculated at point $\xi = 0$, where the variational method always yields a too high value, because this value of ξ is most interesting from the physical standpoint and because the maximum difference is noted at the median plane of the plate ($\xi = 0$).

As long as the value of σ remains small, both methods agree very closely; at a sufficiently large σ (0.2, 0.5, 1.0) there is already less agreement with the difference still within a few percent (at the maximum temperature, in any case). At $\tau = \infty$ the difference $\Delta\theta$ is close to the difference already at $\tau = 2$.

The same comments apply to Fig. 1b, where we note an ideal agreement when $\sigma = 0.02$ and $\sigma = 0.1$. With $\sigma = 1$, the agreement is close up to $\tau = 0.05$ and then worsens to a large discrepancy at $\tau = 0.2$. The agreement improves further till it becomes close to $\tau = \infty$.

The agreement is first close, then worsens, and then improves again for $\sigma = 1$ and for all values of $q^* \approx 2.4674$. The agreement at $\tau = \infty$ improves as q^* decreases, moreover, while for all values of q^* only within a few percent higher than 2.4674 (~10%) this "fluctuation" does not occur. The longer the time interval is, the wider is the discrepancy between any two temperature distributions calculated by each method respectively for the same instant of time, though both distributions are of the same order of magnitude after the longest times considered here. With $q^* = 2.4674$ (this value has been obtained from the equation $1 - q^*/\lambda_0^2 = 0$) the relative error remains almost constant from $\tau = 0.4$ to $\tau = \infty$, while the absolute error decreases slightly.

As has been just said, the agreement is excellent when $\sigma = 0.02$ and 0.1 (Fig. 1c); discrepancies appear only when $\sigma = 1$.

On the other hand, the rather close agreement when $\sigma = 0.02$ and 0.1 is explained by the percentwise effect of this parameter, not sufficiently strong to cause an appreciable change in the thermal conductivity. This is not so when $\sigma = 1$. The preceding analysis applies only to those specific cases, inasmuch as the usually correct values of σ cannot be established without considering also internal heat generation.

The main advantage of the variational method in solving heat conduction problems is that the temperature-dependence of thermal conductivity does not restrict its applicability. In fact, this temperature dependence can be incorporated directly into the variational principle. The finite-differences method is beyond doubt more precise and correct, but the variational method should be considered preferable in terms of machine time – if accuracy within a few percent is adequate. Therefore, the choice of one or another method is dictated by the necessary precision. We also note that, depending on the required

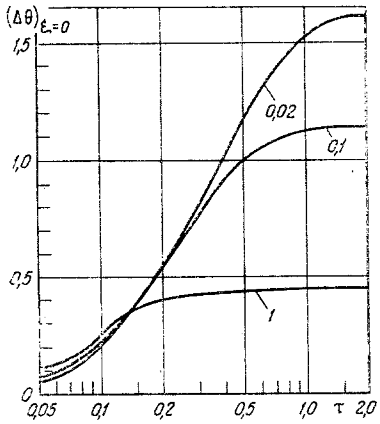

Fig. 2. Difference between temperature distributions based on two different modes of heat generation, as a function of time.

precision, it becomes necessary to test all assumptions made. The rate of heat generation cannot be considered constant, for instance, when the plate is of medium thickness (as in nuclear reactors). Such an assumption would be a rather rough approximation. Let us consider, for example, the following mode of heat generation:

$$q^+ = q^* \cos \frac{\pi}{2} \xi,$$

with $q^* = 20$. If $q^* = 20 = \text{const}$, the extremely large error will render the design of a nuclear reaction unfeasible because of the excessive coolant requirement. The difference between two temperature distributions, both obtained by the finite-differences method, is shown in Fig. 2, for the two said heat generation modes.

We note, in conclusion, that the choice of a method depends on the problem at hand. Particular attention must be paid to the approximations to be made, because they may sometimes reduce the reliability of the results.

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